

# On the PBD-closure of sets containing 3<sup>☆</sup>

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## Abstract

We describe a method used to prove nonexistence of pairwise balanced designs and determine the exact closure of all subsets  $K$  of the set  $\{3, 4, \dots, 22\}$  with  $K \cap \{11, 12, \dots, 22\} \neq \emptyset$  and  $3 \in K$ .  
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## 1. Introduction

Let  $K$  be a set of positive integers. Then a *pairwise balanced design*  $PBD[v, K]$  of order  $v$  with block sizes from  $K$  is a pair  $(V, \mathcal{B})$ , where  $V$  is a finite set (the *point set*) of cardinality  $v$  and  $\mathcal{B}$  is a family of subsets (called *blocks*) of  $V$  which satisfy the following properties:

- (i) if  $B \in \mathcal{B}$ , then  $|B| \in K$ ;
- (ii) every pair of distinct elements of  $V$  occurs in exactly one block of  $\mathcal{B}$ .

A set  $S$  of positive integers is said to be *PBD-closed* if the existence of a  $PBD[v, S]$  implies that  $v$  belongs to  $S$ . Let  $K$  be a set of positive integers and let  $B(K) = \{v \mid \exists PBD[v, K]\}$ . Then  $B(K)$  is a *PBD-closed* set called the *closure* of  $K$ .

In [4] Gronau et al. determined the complete closure of all subsets of the set  $\{3, 4, \dots, 10\}$  which include 3. In this paper we give a generalization of their proofs. To do this the concept of a prestructure is very useful.

Let  $K'$  be a set of positive integers. A *partial pairwise balanced design*  $PPBD[g, K']$  of order  $g$  with block sizes from  $K'$  is a pair  $(G, \mathcal{F})$ , where  $G$  is a point set of cardinality  $g$  and  $\mathcal{F}$  is a family of subsets (called *blocks*) of  $G$  which satisfy the properties:

- (i) if  $F \in \mathcal{F}$ , then  $|F| \in K'$ ;
- (ii) every pair of distinct elements of  $G$  occurs in at most one block of  $\mathcal{F}$ .

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A  $PPBD[v, K']$  with block set  $\mathcal{F}$  is called a  $(K', K)$ -completable prestructure of order  $v$  if there exists a  $PBD[v, K]$  on the same point set and with block set  $\mathcal{B}$ , where  $\mathcal{F} \subseteq \mathcal{B}$ ,  $K' \subseteq K$  and if  $B \in \mathcal{B} \setminus \mathcal{F}$ , then  $|B| \in K \setminus K'$ .

Obviously, nonexistence of a  $(K', K)$ -completable prestructure of order  $v$  implies nonexistence of a  $PBD[v, K]$ . Particularly, if 3 occurs as block length, then this is a good approach to prove nonexistence. Mendelsohn and Rees [5] used the idea to prove necessary conditions for the existence of  $PBD[v, \{3, k\}]$ s with at least one block of size  $k$ .

In this paper we establish the exact closure of all subsets  $K$  of the set  $\{3, 4, \dots, 22\}$  where  $3 \in K$ . The existence or nonexistence of a  $PBD[v, K]$  is determined for all  $v$  if  $K \cap \{11, 12, \dots, 22\} \neq \emptyset$  and  $3 \in K$ . There is no case in doubt. Used construction methods are briefly described.

## 2. General necessary conditions

In this section we introduce several basic necessary conditions.

Let  $\Pi = (V, \mathcal{B})$  be a pairwise balanced design and let  $k_1, k_2, \dots, k_n$  denote the block sizes occurring in  $\Pi$ . We say that  $\beta = (\beta_1, \beta_2, \dots, \beta_n)$  is the *block type* of  $\Pi$  if  $\beta_i$  counts the number of blocks of size  $k_i$ ,  $i = 1, 2, \dots, n$ . For the block type  $\beta$  of  $\Pi$  on  $v$  points the following is true:

**Lemma 1.**

$$\sum_{i=1}^n \beta_i k_i (k_i - 1) = v(v - 1).$$

The proof is done by counting the pairs of points in two ways.

Consider now a point  $x \in V$  and let  $\gamma_i$  count the number of blocks through  $x$  with length  $k_i$ ,  $i = 1, 2, \dots, n$ . The vector  $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$  is called the *point type* of  $x$  in  $\Pi$ . If we want to refer to the point  $x$  we write  $\gamma(x)$ . The point types of  $\Pi$  satisfy the conditions presented in the following lemma.

**Lemma 2.**

$$\sum_{i=1}^n \gamma_i(x) (k_i - 1) = v - 1.$$

The proof is by counting the pairs of elements in  $V$  which contain the point  $x$  in two ways.

If there is a block  $B$  of a certain length  $k$  in  $\Pi$ , then we can often restrict the possible point types for points not on  $B$  by the following lemma, see [4].

**Lemma 3.** Let  $B$  be a block of length  $k$  in  $\Pi$  and let  $\gamma$  be the point type of a point  $x \in V \setminus B$ . Then the following inequality holds:

$$\sum_{i=1}^n \gamma_i(x) \geq k.$$

Let  $(V, \mathcal{B})$  be a  $PBD[v, K]$  and let  $K'$  be a subset of  $K$ . Define  $I(K') = \{i \in \{1, 2, \dots, n\} \mid k_i \in K'\}$ . We introduce the following notation:

$$\beta_{K'} = \sum_{i \in I(K')} \beta_i \text{ the number of blocks in } \mathcal{B} \text{ with sizes in } K',$$

$$\gamma_{K'}(x) = \sum_{i \in I(K')} \gamma_i(x) \text{ the number of blocks through } x \text{ with sizes in } K'.$$

A useful generalization of Lemma 6 in [4] is the following statement:

**Lemma 4.** Let  $K'$  be a subset of  $\{k_1, k_2, \dots, k_n\}$ . Then

$$\beta_{K'}(\beta_{K'} - 1) \geq \sum_{x \in V} \gamma_{K'}(x)(\gamma_{K'}(x) - 1).$$

**Proof.** Count the edges in the following bipartite graph  $G = (V_1 \cup V_2, E)$ . Subset  $V_1$  contains as vertices all pairs  $(B, B') \in \mathcal{B} \times \mathcal{B}$  with  $B \neq B'$ ,  $|B| \in K'$  and  $|B'| \in K'$ . Vertex subset  $V_2$  has as elements all points from point set  $V$ . The edge set is defined as

$$E = \{((B, B'), x) \in V_1 \times V_2 \mid \{x\} = B \cap B'\}.$$

Two blocks intersect in at most one point hence the degree of any vertex in  $V_1$  is at most one. Then  $|V_1| = \beta_{K'}(\beta_{K'} - 1)$  implies that there are at most  $\beta_{K'}(\beta_{K'} - 1)$  edges in  $E$ . Further each vertex  $x \in V_2$  has degree  $\gamma_{K'}(x)(\gamma_{K'}(x) - 1)$  and therefore the number of edges in  $E$  is  $\sum_{x \in V} \gamma_{K'}(x)(\gamma_{K'}(x) - 1)$ .  $\square$

A result on nonexistence is given in the following lemma:

**Lemma 5.** Let  $K$  be a set of positive integers whose smallest element is 3. Suppose that there exists a  $PBD[v, K]$  which contains blocks  $B_h$  and  $B_k$  of sizes  $h$  and  $k$ , respectively, where  $h \geq k$ . Then  $v \geq 2h + k - 2$ .

**Proof.** This is a special case of a theorem proved by Drake and Larson [2] (or see Theorem III.3.1 in [6]).  $\square$

### 3. Necessary conditions for the existence of a prestructure

The aim of this section is to describe some criteria which allow us to prove nonexistence of PBDs just by doing a few simple calculations.

First, we combine necessary conditions on blocks and points to obtain a system of equations and inequalities. Before doing so, let us introduce some further notation.

Let  $\Pi = (V, \mathcal{B})$  be a (partial) pairwise balanced design of order  $v$  and let  $K'$  be a subset of  $K$ , where  $k_{\min}$  is the smallest and  $k_{\max}$  the greatest element of  $K'$ . Let  $m = \lfloor (v - 1)/(k_{\min} - 1) \rfloor$ . Furthermore, denote by

$$x_j = |\{x \in V \mid \gamma_{K'}(x) = j\}| \quad (0 \leq j \leq m),$$

the number of points of  $V$  which lie on exactly  $j$  blocks with lengths from  $K'$ . Let us call the vector  $\mathbf{x} = (x_0, x_1, \dots, x_m)$  the  $K'$ -prestructure type of  $\Pi$ .

**Theorem 6.** *Let  $K$  be a set of positive integers and let  $K'$  be a subset of  $K$ . If there exists a PBD[ $v, K$ ], say  $\Pi$ , then there exists an integer solution  $(\boldsymbol{\beta}; \mathbf{x}) = (\beta_1, \beta_2, \dots, \beta_n; x_0, x_1, \dots, x_m)$  of the following system (S):*

$$\sum_{i=1}^n \beta_i k_i (k_i - 1) = v(v - 1), \tag{1}$$

$$\sum_{j=0}^m x_j = v, \tag{2}$$

$$\sum_{i \in I(K')} k_i \beta_i = \sum_{j=0}^m j x_j, \tag{3}$$

$$\beta_{K'}(\beta_{K'} - 1) \geq \sum_{j=0}^m j(j - 1) x_j, \tag{4}$$

$$x_j \geq 0, \quad j = 0, 1, \dots, m,$$

$$\beta_i \geq 0, \quad i = 1, 2, \dots, n.$$

**Proof.** Let  $\mathbf{x}$  be the  $K'$ -prestructure type and let  $\boldsymbol{\beta}$  be the block type of  $\Pi$ . Eq. (1) is clear by Lemma 1. There is no point  $x$  that lies on more than  $m$  blocks with lengths from  $K'$ , this gives us  $0 \leq \gamma_{K'}(x) \leq m$  and therefore Eq. (2) is true. Eq. (3) follows from counting the points on blocks with block lengths in  $K'$  in two different ways. Lemma 4 and

$$\sum_{x \in V} \gamma_{K'}(x)(\gamma_{K'}(x) - 1) = \sum_{j=0}^m \sum_{\substack{x \in V, \\ \gamma_{K'}(x)=j}} j(j - 1) = \sum_{j=0}^m j(j - 1) \underbrace{\sum_{\substack{x \in V, \\ \gamma_{K'}(x)=j}} 1}_{x_j}$$

establish inequality (4).  $\square$

**Example 7.** We get a PBD[10, {3,4}] by adjoining a new point to every block of a parallel class of the unique  $2 - (9, 3, 1)$  block design.

0	3	6	0	0	0	1	1	1	2	2	2
1	4	7	3	4	5	3	4	5	3	4	5
2	5	8	6	8	7	8	7	6	7	6	8
9	9	9									

This pairwise balanced design has block type  $\beta = (9, 3)$  and  $\{4\}$ -prestructure type  $x = (0, 9, 0, 1)$ . The vector  $(\beta; x)$  satisfies all conditions in (S).

**Remark 8.** In order to prove nonexistence of a pairwise balanced design  $PBD[v, K]$  it is sufficient to show that there exists a subset  $K'$  of  $K$  such that there is no integer solution of system (S). This means there is no  $(K', K)$ -completable prestructure.

### 3.1. Prestructures with all of the blocks of even sizes

We observed during our investigations that a special subset of  $K$ , namely  $K'$  containing all of the even elements of  $K$ , provides best results in determining the nonexistence of  $PBD$ s. Therefore this case will be examined in more detail. From now on it will always be assumed that  $v \notin B(K')$  if  $K' \subset K$ , since  $B(K') \subseteq B(K)$ .

**Theorem 9.** Let  $v$  be a positive integer and  $K$  be a set of positive integers. Suppose that  $v \notin B(K \setminus \{k\})$  whenever  $k \in K$ . Further let

$$\beta_{\min} = \max\{k \in K \mid k \equiv v - 1 \pmod{2}\} + (v \pmod{2}).$$

Let  $K' = \{k \in K \mid k \text{ an even integer}\}$  contain all of the even elements of  $K$ . If there exists a  $PBD[v, K]$ , then the following system (S1) has an integer solution  $(\beta; x)$ .

$$\sum_{i=1}^n \beta_i k_i (k_i - 1) = v(v - 1), \quad (5)$$

$$\sum_{j=0}^m x_j = v, \quad (6)$$

$$\sum_{i \in I(K')} k_i \beta_i = \sum_{j=0}^m j x_j, \quad (7)$$

$$\beta_{K'} (\beta_{K'} - 1) \geq \sum_{j=0}^m j(j - 1) x_j, \quad (8)$$

$$x_j \geq 0, \quad j \equiv v - 1 \pmod{2},$$

$$x_j = 0, \quad j \equiv v \pmod{2}, \quad (9)$$

$$\beta_{K'} \geq \beta_{\min}, \tag{10}$$

$$\beta_i \geq 1, \quad i = 1, 2, \dots, n. \tag{11}$$

**Proof.** Theorem 6 implies that there is a solution  $(\beta; \mathbf{x})$  of system (S). Hence Eqs. (5)–(7), and inequality (8) are satisfied by this solution. This solution satisfies also Eq. (9) since Lemma 2 implies  $\forall x \in V$  that

$$\sum_{i \in I(K \setminus K')} \gamma_i(x)(k_i - 1) + \sum_{i \in I(K')} \gamma_i(x)(k_i - 1) = v - 1$$

and this indicates that

$$\sum_{i \in I(K')} \gamma_i(x)(k_i - 1) \equiv \sum_{i \in I(K')} \gamma_i(x) = \gamma_{K'}(x) \equiv v - 1 \pmod{2}.$$

Thus the number of blocks with sizes from the set  $K'$  on any point must be congruent to  $v - 1$  modulo 2. Hence  $x_j = 0$  if  $j \equiv v \pmod{2}$ .

If  $v$  is even, then through every point on a block of odd length goes at least one block of even length, since  $x_0 = 0$ . Blocks through distinct points on one block are distinct hence  $\beta_{K'} \geq k$  is true for all odd  $k \in K$ . Suppose  $v$  is odd, there exists at least one block of length  $k_{\max}$  and through every point of this block goes at least one other block of even length, since  $x_1 = 0$ . Thus there are at least  $k_{\max} + 1$  blocks of even sizes. Hence  $\beta_{K'}$  satisfies inequality (10).

Moreover  $\beta_i \geq 1$  since by assumption  $v \notin B(K \setminus \{k_i\})$ .  $\square$

**Corollary 10.** *If  $\beta_{K'}$  is even, then replace (8) by the following:*

$$\beta_{K'}(\beta_{K'} - 2) \geq \sum_{j=0}^m j(j - 1)x_j.$$

**Proof.** Let  $B$  be a block of even size. Every point on  $B$  lies on an odd (resp. even) number of even blocks. Hence the total number of even blocks which intersect  $B$  is an even integer. Then there exists at least one block that does not intersect  $B$ , since  $\beta_{K'}$  is even. In other words, every even block intersects at most  $\beta_{K'} - 2$  other even blocks. Finally, the arguments in the proof of Lemma 4 and Theorem 6 establish the corollary.  $\square$

It is natural to ask whether it is possible to find a solution which is easy to compute. The next two lemmas will establish that a special vector  $(\beta^*; \mathbf{x}^*)$  solves system (S1) whenever there is a solution of (S1). The investigation of this vector is motivated by the fact that in all of the existence cases we found a *PBD* with a prestructure corresponding to  $(\beta^*; \mathbf{x}^*)$ .

For the sake of brevity we write  $\sum_{K'}^\beta$  instead of  $\sum_{i \in I(K')} k_i \beta_i$ .

**Lemma 11.** Suppose that  $(\beta; \mathbf{x}')$  is solution of (S1) and  $j^*$  is the greatest integer with  $j^* \leq \lfloor \Sigma_{K'}^\beta / v \rfloor$  and  $j^* \equiv v - 1 \pmod{2}$ . Then  $(\beta; \mathbf{x}^* = (x_0^*, x_1^*, \dots, x_m^*))$ , where

$$x_j^* = \begin{cases} 0, & j \neq j^*, j \neq j^* + 2, \\ \frac{(j^* + 2)v - \Sigma_{K'}^\beta}{2}, & j = j^*, \\ \frac{\Sigma_{K'}^\beta - j^*v}{2}, & j = j^* + 2 \end{cases}$$

is also a solution of (S1).

**Proof.** Clearly (5) is satisfied. Observe that  $\mathbf{x}^*$  is well defined, since  $j^* \geq 0$  if  $v$  is odd,  $j^* \geq 1$  if  $v$  is even and  $x_{j^*+2} = 0$  if  $j^* + 2 > m$ . Moreover  $\mathbf{x}^*$  satisfies Eqs. (6) and (7):

$$\sum_{j=0}^m x_j^* = x_{j^*}^* + x_{j^*+2}^* = \frac{j^*v + 2v - \Sigma_{K'}^\beta + \Sigma_{K'}^\beta - j^*v}{2} = v$$

and

$$\sum_{j=0}^m j x_j^* = \frac{j^*(j^* + 2)v - j^* \Sigma_{K'}^\beta + (j^* + 2) \Sigma_{K'}^\beta - (j^* + 2)j^*v}{2} = \Sigma_{K'}^\beta.$$

We now have to prove that  $\mathbf{x}^*$  also satisfies inequality (8). It is easy to verify that  $\mathbf{x}^*$  and  $j^*$  are the unique vector and number, respectively, which satisfy Eqs. (6) and (7) and in addition:

- (i)  $j^* \in \{1, 2, \dots, m\}$ ,  $j^* \equiv v - 1 \pmod{2}$ ,
- (ii)  $x_{j^*}^* > 0$ ,  $x_{j^*+2}^* \geq 0$  and  $x_j = 0$  if  $j \neq j^*$  or if  $j \neq j^* + 2$ .

Let  $x'_{j^-}$  be the first nonzero component of  $\mathbf{x}'$  and  $x'_{j^+}$  the last nonzero component. If  $j^- - 2 \geq j^+$  holds, then  $\mathbf{x}' = \mathbf{x}^*$  and therefore  $(\beta; \mathbf{x}^*)$  is solution of system (S1). Otherwise define a new vector  $\mathbf{x}^1$  by

$$x_j^1 = \begin{cases} x'_j - 1, & j = j^-, j = j^+, \\ x'_j + 1, & j = j^- + 2, j = j^+ - 2, \\ x'_j, & \text{else.} \end{cases}$$

Now  $\beta_{K'}(\beta_{K'} - 1) > \sum_{j=0}^m j(j-1)x_j^1$  since  $\sum_{j=0}^m j(j-1)x'_j \geq 8 + \sum_{j=0}^m j(j-1)x_j^1$ . By its definition  $\mathbf{x}^1$  satisfies (6), (7) and (9) hence  $(\beta; \mathbf{x}^1)$  is solution of (S1). Determine  $j^-$  and  $j^+$  of  $\mathbf{x}^1$  and (pairwise distinct) solution vectors  $\mathbf{x}^2, \mathbf{x}^3, \dots$ . After finitely many steps we obtain a solution  $\mathbf{x}^t$  of (S1) which has at most two nonzero components and these occur in consecutive positions (with indices of the same parity). Hence  $\mathbf{x}^t = \mathbf{x}^*$  which completes the proof.  $\square$

**Remark 12.** If  $\beta$  is fixed, then  $\mathbf{x}^*$  minimizes  $\sum_{j=0}^m j(j-1)x_j$  over all  $\mathbf{x}$  satisfying (S1).

In order to have as few intersections between blocks of even sizes as possible, we want to have  $\sum_{i \in I(K')} k_i \beta_i$  as small as possible. If the number of blocks is fixed, then this means that we want as many blocks of smallest even block size as possible, and as few blocks of other even block lengths as possible.

**Lemma 13.** *Let  $v$  be a positive integer. Suppose that  $K'$  is the subset of  $K$  which contains all of the even elements. If a  $PBD[v, K]$  with  $\beta$  blocks of even lengths exists, then  $(\beta^*; \mathbf{x}^*)$  is an integer solution of system (S1), where  $\beta^*$  is a solution of the following optimizing problem (O) :*

$$(O) \left\{ \begin{array}{l} \text{minimize } \sum_{i \in I(K')} k_i \beta_i \text{ under the conditions :} \\ \sum_{i=1}^n \beta_i k_i (k_i - 1) = v(v - 1), \\ \sum_{i \in I(K')} k_i \beta_i \geq v \quad (\text{only if } v \text{ is even}), \\ \beta = \sum_{i \in I(K')} \beta_i, \\ \beta_i \geq 1, \quad i = 1, 2, \dots, n \end{array} \right.$$

and  $\mathbf{x}^* = (0, \dots, 0, x_{j^*}^*, 0, x_{j^*+2}^*, 0, \dots, 0)$  is defined as in Lemma 11 where  $j^*$  is the greatest integer with  $j^* \leq \lfloor \Sigma_{K'}^{\beta^*} / v \rfloor$  and  $j^* \equiv v - 1 \pmod{2}$ .

**Proof.** Eq. (5) is true by the definition of  $\beta^*$ . Eqs. (6), (7) and (9) are satisfied by the definition of  $\mathbf{x}^*$ .

Again one must show that inequality (8) holds. Let  $\beta'$  be the block type of the  $PBD[v, K]$  and let  $\mathbf{x}'^* = (0, \dots, 0, x_{j'^*}^{'*}, 0, x_{j'^*+2}^{'*}, 0, \dots, 0)$  belong to  $\beta'$ , where  $j'^* \equiv v - 1 \pmod{2}$  is the greatest integer satisfying  $j'^* \leq \lfloor \Sigma_{K'}^{\beta'} / v \rfloor$ . Then  $(\beta'; \mathbf{x}'^*)$  is a solution of (S1) and

$$\sum_{j=0}^m j x_j'^* = \sum_{i \in I(K')} k_i \beta_i' \geq \sum_{i \in I(K')} k_i \beta_i^* = \sum_{j=0}^m j x_j^*.$$

This implies that  $j'^* \geq j^*$  and therefore  $\sum_{j=0}^m j(j-1)x_j'^* \geq \sum_{j=0}^m j(j-1)x_j^*$ .

Then

$$\beta_{K'}^* (\beta_{K'}^* - 1) = \beta_{K'}' (\beta_{K'}' - 1) \geq \sum_{j=0}^m j(j-1)x_j'^* \geq \sum_{j=0}^m j(j-1)x_j^*.$$

Inequalities (10) and (11) are true since  $\beta^*$  is solution of (O).  $\square$

**Remark 14.** If for  $\beta = \beta_{\min}$  up to  $\beta = \beta_{\max} := \lfloor m v / k_{\min} \rfloor$  the vector  $(\beta^*; \mathbf{x}^*)$  derived as in Lemma 13 is not a solution of (S1), then there exists no  $PBD[v, K]$ . We call  $(\beta^*; \mathbf{x}^*)$  the  $(K', \beta)$ -optimal vector.

Let us give the following example, to explain how nonexistence can be proved.

**Example 15.** Let  $v = 74$  and  $K = \{3, 10, 11\}$ . Take  $K' = \{10\}$ , then  $m = \lfloor \frac{73}{9} \rfloor$  and  $\beta_{\min} = 11$ . Start with  $\beta = 11$ , then  $j^* = 1$ , since  $j^* \leq \lfloor \frac{110}{74} \rfloor$  and  $j^*$  is odd. Compute  $x_1^* = (3 \times 74 - 110)/2 = 56$  and  $x_3^* = (110 - 74)/2 = 18$  to obtain the  $(\{10\}, 11)$ -optimal



vector  $(\beta^*; \mathbf{x}^*) = (717, 11, 1; 0, 56, 0, 18, 0, 0, 0, 0, 0)$  that satisfies (S1). Therefore there could be a  $PBD[74, \{3, 10, 11\}]$  with 11 blocks of size 10. (Check that there is no further solution of (S1) with 11 blocks and have a look at Proposition 29.)

Let us now consider  $\beta = 12$ . The vector  $(\beta^*; \mathbf{x}^*) = (702, 12, 1; 0, 51, 0, 23, 0, 0, 0, 0, 0)$  is not a solution of (S1), since inequality (8)  $132 \not\geq 6 \times 23$  is not satisfied. Hence there exists no  $PBD[74, \{3, 10, 11\}]$  with 12 blocks of size 10. The same holds for  $13 \leq \beta \leq 20$ .

Finally set  $\beta = 21$ . The vector  $(\beta^*; \mathbf{x}^*) = (567, 21, 1; 0, 6, 0, 68, 0, 0, 0, 0, 0)$  solves system (S1). We point out that every  $PBD[74, \{3, 10, 11\}]$  has this prestructure type.

Note, that in Example 15 the optimizing problem (O) was easy to solve, but in other cases there are further congruential conditions on the number of blocks to consider. Another example may help to clarify this.

**Example 16.** Let  $v = 36$ ,  $K = \{3, 8\}$ . There exists no optimal solution of (O) with  $\beta = \beta_{\min} + 1 = 4$ , since the number of blocks of size 8 must be congruent 0 modulo 3 (see Lemma 1).

Let  $v = 36$ ,  $K = \{3, 8, 10\}$ ,  $\beta = 5$ . Then  $\beta^* = (152, 3, 2)$ , since  $\beta^* = (\beta_1, 4, 1)$  is not possible for the same reason.

### 3.2. Adding blocks of odd lengths to the prestructure

In this section it is assumed that there are both blocks of odd and even lengths. Let  $v$  be a positive even integer and  $k$  be the largest odd block size of  $K$ . To make sure that it is possible to include a block of size  $k$  into a prestructure whose prestructure type is derived in (S1) we establish the following claim.

**Proposition 17.** Let  $v$  be a positive even integer and  $k \in K$  be the largest odd block size. Assume the existence of a  $PBD[v, K]$ , say  $\Pi$ . Let  $K'$  be the set of all of the even block lengths and let  $\mathbf{x}$  be the  $K'$ -prestructure type of  $\Pi$ . Further let  $\tilde{x} = \max\{0, k - \lfloor(\beta_{K'} - k)/2\rfloor\}$ . Then  $x_1 \geq \tilde{x}$ .

**Proof.** Let  $B$  be a block in  $\Pi$  of size  $k$ . Let us designate the number of blocks with even lengths which intersect  $B$  by  $\beta_B$ . Furthermore, we denote by  $b_1, b_2$ , respectively, the number of points on  $B$  which lie on exactly one or at least three blocks of sizes from  $K'$ . Then  $b_1 + b_2 = k$ , since there is no point in  $B$  which lies in two even blocks, see proof of Theorem 9. Moreover  $\beta_B \geq b_1 + 3b_2$ . It follows that  $b_1 \geq k - (\beta_B - k)/2$ . Thus, observing that  $x_1 \geq b_1$  and  $x_1 \geq 0$  concludes the proof.  $\square$

**Example 18.** Let  $K = \{3, 6, 11\}$ ,  $K' = \{6\}$  and  $v = 32$ . Assume that  $\Pi$  is a  $PBD[v, K]$ . Remark 14 implies  $11 \leq \beta_{K'} \leq 26$ . There is no point in  $\Pi$  that lies on more than 3 blocks of size 6 (see Example 28). Hence  $x_1 + x_3 = 32$  and  $6\beta_{K'} = x_1 + 3x_3 = 96 - 2x_1$ .

This implies  $96 - 6\beta_{K'} = 2x_1 \geq 2\tilde{x} \geq 2(11 - (\beta_{K'} - 11)/2)$  and therefore  $\beta_{K'} \leq 12$ . Thus, we only need to consider  $\beta_{K'} = 11, 12$ .

A similar proposition can be stated if  $v$  is odd. Here look at the largest even block.

**Proposition 19.** *Let  $v$  be a positive odd integer and  $k \in K$  be the largest even block size. Assume the existence of a  $PBD[v, K]$ , say  $\Pi$ . Let  $K'$  be the set of all of the even block lengths and let  $\mathbf{x}$  be the  $K'$ -prestructure type of  $\Pi$ . Further let  $\tilde{x} = \max\{0, k - \lfloor (\beta_{K'} - k - 1)/2 \rfloor\}$ . Then  $x_2 \geq \tilde{x}$ .*

**Proof.** Recall, that every point of  $\Pi$  lies on an even number of even blocks. Hence, looking at points on a fixed block of size  $k$  which lie on exactly two or at least four even blocks gives the required inequality.  $\square$

With respect to Propositions 17 and 19 add to (S1) the following inequality:

$$x_{1+(v \bmod 2)} \geq \tilde{x} \tag{12}$$

to obtain system  $(\widetilde{S1})$ .

**Theorem 20.** *Let  $v$  be a positive integer and  $K$  a set of positive integers. Assume the existence of a  $PBD[v, K]$ . Then system  $(\widetilde{S1})$  has an integer solution  $(\boldsymbol{\beta}, \mathbf{x})$ .*

**Proof.** Use Theorem 9, Propositions 17 and 19 to establish the theorem.  $\square$

**Remark 21.** To see whether there exists a solution of  $(\widetilde{S1})$  or not, replace  $v$  by  $\tilde{v} := v - \tilde{x}$  and  $\Sigma_{K'}^\beta$  by  $\Sigma_{K'}^\beta - (1 + (v \bmod 2)) \cdot \tilde{x}$  in Lemma 11.

**Example 22.** Let  $v = 74$ ,  $K = \{3, 10, 11\}$  and  $K' = \{10\}$  as in Example 15. Set  $\beta = 22$ . Then  $\tilde{x} = 6$ . Put  $\tilde{v} = 68$  and  $\Sigma_{K'}^\beta = 214$  to obtain the vector  $(552, 22, 1; 0, 0, 0, 63, 0, 5, 0, 0, 0)$  from Lemma 11 that leads to an optimal vector  $(\boldsymbol{\beta}^*, \mathbf{x}^*) = (552, 22, 1; 0, 0 + 6, 0, 63, 0, 5, 0, 0, 0)$ . But this implies that there is no  $PBD[74, \{3, 10, 11\}]$  with 22 blocks of size 10, since inequality (8) is not satisfied  $(22 \times 20 \not\geq 6 \times 63 + 20 \times 5)$ .

### 3.3. Uncompletable prestructures

In this section we consider solution vectors which lead to partial pairwise balanced designs but not to a completable prestructure. The propositions stated are often just simple observations, but they rule out several parameter sets (see Appendix A).

**Example 23.** Let  $K = \{3, 6\}$ ,  $K' = \{6\}$  and let  $v = 22$ . There exists a unique solution of  $(\widetilde{S1})$ :  $(57, 4; 0, 21, 0, 1, 0)$ . But  $22 \notin B(\{3, 6\})$ . Suppose that there is  $PBD[v, K]$ , then

there exists a point  $p \in V$  that lies on exactly 3 blocks of length 6, say  $D_1, D_2, D_3$ , and on exactly 3 blocks of length 3. Let  $\{p, x, y\}$  be such a block of size 3, then  $x$  and  $y$  are not points on  $D_1, D_2$  or  $D_3$ , respectively. Thus  $\{x, y\} \subset D_4$  the fourth block of length 6, contradicting the fact that every pair of points occurs in exactly one block.

This example will be generalized by the following proposition:

**Proposition 24.** *Let  $v$  be a positive even integer, let  $K$  be a set of positive integers and  $K'$  be the subset of  $K$  containing all of the even elements. Now let  $(\beta; \mathbf{x})$  be a solution of system  $(\widetilde{S1})$ . Further let us denote  $\sum_{i \in I(K')} \beta_i$  by  $\beta$ . If  $x_{\beta-1} = 1$ , then there exists no  $(K', K)$ -completable prestructure and hence no  $PBD[v, K]$  with  $K'$ -prestructure type  $\mathbf{x}$ .*

**Proof.** The proof is done by the same argument as in the example above. Note that  $x_{\beta-1} = 1$  implies  $x_1 = v - 1$ , since  $x_2 = 0$ .  $\square$

The following proposition is obvious.

**Proposition 25.** *Let  $v$  be a positive even integer. Suppose that there exists a  $PBD[v, K]$  and suppose that 3 is the smallest element in  $K$ . Further let  $K'$  be the subset of  $K$  containing all of the even elements. If  $\beta_{K'} = 3$ , then  $|K'| = 1$ .*

**Proof.** Every block of length 3 intersects the three blocks of lengths from  $K'$ . Thus, these blocks have same size.  $\square$

The next claim follows from observing that  $m$  defined in the beginning of Section 3 as  $\lfloor v - 1/(k_{\min} - 1) \rfloor$  is only an upper bound for the maximal number of blocks of lengths from  $K'$  a point lies on. We say  $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$  is a *possible point type* if  $\gamma$  satisfies the equation in Lemma 2 and  $\sum_{i=1}^n \gamma_i \geq k_i$  whenever  $\gamma_i = 0$ ; the latter corresponds to Lemma 3. Define  $\gamma_{K'} = \sum_{i \in I(K')} \gamma_i$ . It should be remarked that the same notation is used as for point type, since every point type vector is also a possible point type.

**Proposition 26.** *Let  $v$  be a positive integer and let  $K'$  be a subset of block size set  $K$ . Further let  $m' = \max \gamma_{K'}$  over all possible point types. If  $(\beta; \mathbf{x})$  is a solution of system  $(\widetilde{S1})$  and  $x_j > 0$  for some  $j > m'$ , then there exists no  $(K', K)$ -completable prestructure with  $K'$ -prestructure type  $\mathbf{x}$ .*

**Proof.** Assume the existence of a  $PBD[v, K]$ . Then Lemmas 2 and 3 imply that there is no point that lies on more than  $m'$  blocks of lengths from  $K'$ .  $\square$

**Remark 27.** The integer  $m$  is only used in the sense of the maximal number of blocks from the prestructure a point can lie on. Therefore it is possible to replace  $m$  by  $m'$  in all of the theorems above.

**Example 28.** Let  $K = \{3, 6, 11\}$  and  $v = 32$ . For brevity write  $3^3 6^5$  instead of  $\gamma = (3, 5, 0)$ . Possible point types are  $3^3 6^1 11^2$ ,  $3^3 6^3 11^1$ ,  $3^8 6^1 11^1$ ,  $3^8 6^3$  and  $3^{13} 6^1$ .  $3^3 6^5$  is not a possible point type since it satisfies the equation from Lemma 2 but  $3 + 5 < 11$ . Hence  $m' = 3 < m = 5$ .

**Proposition 29.** Let  $v$  be a positive even integer and let  $K'$  be the subset of  $K$  which contains all of the even elements. Suppose that there exists no solution of  $(\widetilde{S1})$  with  $\beta_{K'} \equiv 5 \pmod{6}$  and  $x_1 + x_3 < \tilde{v}$ . If  $(\beta; x)$  is a solution of system  $(\widetilde{S1})$  with  $\beta_{K'} \equiv 5 \pmod{6}$  and  $\beta_{K'}(\beta_{K'} - 1) - 6 < \sum_{j=1}^m j(j-1)x_j$ , then there exists no  $(K', K)$ -completable prestructure with  $K'$ -prestructure type  $x$ .

**Proof.** Let  $\Pi$  be a  $PBD[v, K]$  with  $\beta_{K'} \equiv 5 \pmod{6}$ . Suppose that each block of even size intersects every other block with length from  $K'$ . Since  $x_j = 0$  for all  $j > 3$ , by the proof of Lemma 4 we know that  $\beta_{K'}(\beta_{K'} - 1) = 6x_3$ . Hence 6 divides  $\beta_{K'}(\beta_{K'} - 1) \equiv 2 \pmod{6}$ , a contradiction. Thus there is one and therefore three blocks of even lengths which intersect at most  $\beta_{K'} - 3$  other blocks from the prestructure. Now  $(\beta_{K'} - 3)(\beta_{K'} - 1) + 3(\beta_{K'} - 3) = \beta_{K'}(\beta_{K'} - 1) - 6 \geq 6x_3$ .  $\square$

**Example 30.** Let  $K = \{3, 11, 16\}$  and  $v = 140$ . There is no solution of  $(\widetilde{S1})$  with  $\beta_{K'} \geq 12$  or  $\beta_{K'} = 11$  and  $x_j > 0$  for some  $j > 3$ . Then there is a unique solution  $(\beta_1, \beta_2, 11; 0, 122, 0, 18, 0, 0, 0)$  which satisfies inequality (8):  $110 \geq 108$ . But  $110 - 6 < 108$ , hence  $140 \notin B(\{3, 11, 16\})$ .

**Proposition 31.** Let  $v$  be a positive even integer and  $K$  be a set of positive integers, including 3. Let  $K'$  be the subset of  $K$  which contains all of the even integers. Suppose that  $(\beta; x)$  is a solution of system  $(\widetilde{S1})$  with  $x_1 = v$  and  $\beta_{K'} = 4$ . Let us denote the four (not necessarily distinct) integers of  $K'$  for which  $\beta_i > 0$  holds by  $l_1, \dots, l_4$ . If  $l_1 l_2 + l_3 l_4$  is not equal to the number of blocks of length 3, then there exists no  $(K', K)$ -completable prestructure with  $K'$ -prestructure type  $x$ .

**Proof.** Assume the existence of a  $PBD[v, K]$  with four blocks  $B_1, \dots, B_4$  of size  $l_1, \dots, l_4$ , respectively. Every block of length 3 has either two points on  $B_1$  and  $B_2$  or on  $B_3$  and  $B_4$ , but not both. There are exactly  $l_1 l_2$  blocks of length 3 with two points on  $B_1$  and  $B_2$  and exactly  $l_3 l_4$  blocks of length 3 with two points on  $B_3$  and  $B_4$ .  $\square$

**Example 32.** Let  $K = \{3, 14, 20\}$  and  $v = 68$ . The only solution of  $(\widetilde{S1})$  is  $(572, 2, 2; 0, 68, 0, 0)$ . Let  $B_1$  and  $B_2$  be the blocks of length 20,  $B_3$  and  $B_4$  of size 14. Then  $20^2 + 14^2 = 596 \neq 572$  and therefore  $68 \notin B(\{3, 14, 20\})$ .

**Remark 33.** Note that the proof of the last proposition implies that either  $l_1 = l_2 = l_3 = l_4$  or  $l_1 = l_2 = l_3$ , since these block sizes must satisfy  $l_1 l_2 + l_3 l_4 = l_{\pi(1)} l_{\pi(2)} + l_{\pi(3)} l_{\pi(4)}$  for every permutations  $\pi$  of the indices.

## 4. Construction of PBDs

### 4.1. Recursive constructions

We used various recursive constructions and existence results from the *CRC Handbook of Combinatorial Designs* [6]. In particular, recursive constructions are obtained from Chapter III.2 *PBDs: Recursive Constructions* (pp. 193–202) and we also used the known existence results from Section III.1.3 *Some Families of Group Divisible Designs* (pp. 189–191). All necessary definitions, methods and examples are given there.

Let  $K$  and  $G$  be sets of positive integers. A *group divisible design* (*GDD*) of order  $v$  is a triple  $(V, \mathcal{G}, \mathcal{B})$ , where  $V$  is a finite set of cardinality  $v$ ,  $\mathcal{G}$  is a partition of  $V$  into parts (*groups*) whose sizes lie in  $G$  and  $\mathcal{B}$  is a family of subsets (*blocks*) of  $V$  which satisfy the properties:

- (i) if  $B \in \mathcal{B}$ , then  $|B| \in K$ ;
- (ii) every pair of distinct elements of  $V$  occurs in exactly one block or one group, but not both.

*GDDs* are useful for constructing *PBDs*. By regarding groups to be blocks a *GDD* can be viewed as a  $PBD[v, K \cup G]$ . For reducing the set of block sizes of a *PBD* an operation called *breaking up blocks* can be applied. Let  $\Pi$  be a  $PBD[v, K]$  and  $k \in K$ . Suppose that there exists a  $PBD[k, K \setminus \{k\}]$ , say  $\Pi'$ . By replacing every block of size  $k$  of  $\Pi$  by the blocks of  $\Pi'$  in each case defined on the point set of the block we obtain a  $PBD[v, K \setminus \{k\}]$ .

### 4.2. The hill-climbing algorithm

For the remainder of this paper we consider only sets  $K$  which include 3. In order to start recursive constructions one needs *PBDs* of small order as “ingredients”. These ingredients are produced by hill climbing which is a very successful random algorithm for constructing combinatorial designs with block length 3. For an introduction to the hill-climbing method see [3,7].

Given a completable prestructure we used the hill-climbing approach to complete the missing blocks of length 3. A variation of the hill-climbing algorithm (described in [4]) is applied.

Let  $\mathcal{P} = (V, \mathcal{F})$  be a  $(K', K)$ -completable prestructure. Define the set  $F$  of forbidden pairs as  $F = \{\{x, y\} \mid \exists B \in \mathcal{F} \text{ such that } \{x, y\} \in B\}$ . Let  $G$  be a graph with vertex set  $V$  and edge set  $E = \{\{x, y\} \mid x, y \in V, x \neq y, \{x, y\} \notin F\}$ . We search for a decomposition of  $G$  into triangles. Let  $T$  denote the set of triangles we have found so far. Starting with input  $v$ ,  $F$  and  $G$ , the algorithm works as follows:

*Step 0:*  $T := \emptyset$

*Step 1:* Pick randomly  $p \in V$  such that  $d_G(p) > 0$ . If for all  $p \in V$  the degree in  $G$  is 0, then Goto **End**.

Step 2: Pick randomly two edges  $xp$ ,  $yp$  from  $E$ . If  $xy \in F$ , then Goto **Step 1**.

Step 3: (a) There is no triangle  $xyz \in T$ ,  $z \in V$ . Then take the triangle  $xyp$ .

$$T := T \cup \{xyp\}$$

$$E := E \setminus \{xp, yp, xy\}$$

(b) There is a triangle  $xyz \in T$ ,  $z \in V$ . Then delete that triangle and add  $xyp$ .

$$T := T \cup \{xyp\} \setminus \{xyz\}$$

$$E := E \cup \{xz, yz\} \setminus \{xp, yp\}$$

Goto **Step 1**.

**End**

An important question is: How can one find a completable prestructure?

### 4.3. Determining a completable prestructure

The more intersections between blocks of lengths greater than 3, the more complicated the prestructure becomes. We used the following approach to determine a  $(K', K)$ -completable prestructure. First compute the  $(K', \beta_{K'})$ -optimal solution of  $(\widetilde{S1})$  with  $\beta_{K'}$  as small as possible, say  $(\beta^*; \mathbf{x}^*)$ . This gave us information about the number of blocks of each length and about intersections between these blocks.

Suppose that there exists a partial pairwise balanced design with all blocks of even lengths, whose prestructure type is  $\mathbf{x}^*$ . Let us now regard points to be blocks and blocks to be points, this means that we transpose the incidence matrix. A  $PPBD[\beta_{K'}, H]$ , say  $\Sigma$ , is obtained where  $H = \{j \mid x_j^* > 0\}$ . In particular, there are exactly  $x_j$  blocks of length  $j$ . Furthermore, let  $K' = \{k'_1, \dots, k'_{|K'|}\}$  with  $k'_i < k'_{i+1}$  for  $i = 1, \dots, |K'| - 1$  and  $\beta'_i$  denote the number of blocks of size  $k'_i$ . Then there are  $\beta'_1$  points which occur in at most  $k'_1$  blocks of  $\Sigma$ , there are  $\beta'_1 + \beta'_2$  points which occur in at most  $k'_1 + k'_2$  blocks of  $\Sigma$  and so on.

We now try to construct a  $PPBD[\beta_{K'}, H]$  where all the properties are satisfied. It turns out that in all cases considered  $H = \{1, 3, 5\}$  or  $H = \{2, 4, 6\}$  and the number of blocks of length 2 or 3 is not zero. We do not care about blocks of length one. The blocks of sizes greater than 3 are constructed by backtracking, which is quick if there are enough blocks of size 2 or 3. These blocks are inserted again by hill-climbing. (It is easy to find a hill-climbing-like algorithm for block size 2.) If this is done add without problem blocks of size 1 and transpose the incidence matrix to obtain a  $PPBD[v, K']$ .

Adding blocks of odd lengths is easy, with a few exceptions. We use particularly points which lie on only one or no block of sizes from  $K'$ .

**Example 34.** Let us return to Example 7,  $v = 10$  and  $K = \{3, 4\}$ . The optimal solution is:  $(9, 3; 0, 9, 0, 1)$ . We search for a  $PPBD[3, \{1, 3\}]$  on the point set  $\{a, b, c\}$  with 9

blocks of size one and 1 block of size 3. Here is one:

---

$a$	$a$	$a$	$b$	$b$	$b$	$c$	$c$	$c$	$a$
									$b$
									$c$

---

which leads to the original prestructure by changing points and blocks.

Sometimes, if the structure of the  $PPBD[\beta_{K'}, H]$  is very special, constructing the prestructure with the hill-climbing approach was not successful. But the main idea, changing points and blocks, leads to a result.

**Example 35.** Let  $v = 182$ ,  $K = \{3, 16, 17\}$ , then  $(4125, 33, 1; 0, 9, 0, 173, 0, 0)$  is a solution of  $(S1)$ . A  $PPBD[33, \{1, 3\}]$  is obtained from a 3- $GDD$  on 11 groups of size 3. Eight groups are considered to be blocks. The nine points from remaining groups are blocks of length 1. These  $8 + 9$  blocks are the points of the block of length 17.

A list with all prestructures used can be found at:

<ftp://ftp.math.uni-rostock.de/pub/members/mgruttmm/prestruc.htm>

## 5. The closure of sets containing 3

Let  $\mathbb{N}$  denote the set of all positive integers,  $\mathbb{N}_{1(2)}$  denote the subset of all odd integers and  $\mathbb{N}_{0,1(3)}$  denote the set of positive integers which are congruent 0 or 1 (mod 3). We call two integer sets  $N_1$  and  $N_2$  *coterminal* when there is an integer  $n$  such that  $\{x \in N_1 \mid x \geq n\} = \{x \in N_2 \mid x \geq n\}$ . It is well known that  $B(\{3\}) = \mathbb{N}_{1,3(6)}$ . By Wilson's theory [8] there are three cases for the closure of a set  $K$  which includes 3 and  $K$  is not a subset of  $\mathbb{N}_{1,3(6)}$ .

1. If  $K$  contains only odd integers and is not a subset of  $\mathbb{N}_{1,3(6)}$ , then  $B(K)$  is coterminal with  $\mathbb{N}_{1(2)}$ .
2. If  $K$  contains a positive even integer and all members of  $K$  lie in  $\mathbb{N}_{0,1(3)}$ , then  $B(K)$  is coterminal with  $\mathbb{N}_{0,1(3)}$ .
3. If  $K$  contains a positive even integer and is not a subset of  $\mathbb{N}_{0,1(3)}$ , then  $B(K)$  is coterminal with  $\mathbb{N}$ .

### 5.1. The closure of sets containing only odd integers including 3

**Lemma 36.** Let  $k \equiv 5 \pmod{6}$  be a positive integer, then

$$B(\{3, k\}) = \mathbb{N}_{1(2)} \setminus \{v \in \mathbb{N} \mid v \equiv 5 \pmod{6}, v < 2k + 1, v \neq k\}.$$

**Proof.** We need only consider the case  $v \equiv 5 \pmod{6}, v \neq k$ , since  $B(\{3\}) \subset B(\{3, k\})$ . By a theorem due to Colburn et al. [1] (Theorem III.1.24 in [6]) there exists a 3- $GDD$

of type  $k^1 1^{v-k}$  if the following conditions are satisfied:

1.  $k \equiv 1 \pmod{2}$ ,
2.  $k + (v - k) \equiv 1 \pmod{2}$ ,
3.  $v - k \geq k + 1$ ,
4.  $\binom{v-k}{2} + k(v - k) + \binom{1}{2} k^2 \equiv 0 \pmod{3}$ .

It is easy to check that all conditions are satisfied, if  $v \equiv k \equiv 5 \pmod{6}$  and  $v \geq 2k + 1$ . In this case there exists 3-GDD which is a  $PBD[v, \{3, k\}]$ . If  $v < 2k + 1$  ( $v \neq k$ ), then Lemma 5 implies that  $v \notin B(\{3, k\})$ .  $\square$

**Theorem 37.** *Let  $K$  contain only odd integers (including 3) and let  $k$  be the smallest integer in  $K$  with  $k \equiv 5 \pmod{6}$ , then*

$$B(K) = K \cup \mathbb{N}_{1,3(6)} \cup \{v \in N \mid v \equiv 5 \pmod{6}, v \geq 2k + 1\}.$$

**Proof.** Again we need only consider the case  $v \equiv 5 \pmod{6}, v \neq k$ . If  $v \geq 2k + 1$ , then Lemma 36 implies  $v \in B(\{3, k\}) \subseteq B(K)$ . Suppose that there exists a  $PBD[v, K]$  with  $v < 2k + 1$  and  $v \notin K$ , then there are two blocks of length  $k_1, k_2$  with  $k_1 \equiv 5 \pmod{6}$ . The following inequality is satisfied:

$$2k_1 + k_2 - 2 \geq 2k_1 + 1 \geq 2k + 1 > v.$$

But by Lemma 5 we have  $v \geq 2k_1 + k_2 - 2$ , a contradiction.  $\square$

### 5.2. The closure of sets containing 3 and an even integer and all elements lie in $\mathbb{N}_{0,1(3)}$

We used a program written in C to verify whether a solution of  $(\widetilde{S1})$  exists or not. If a solution exists, then the program tries to find a recursive construction or a suitable prestructure to construct the  $PBD$ . Otherwise it is checked by the program or by hand that no solution leads to a completable prestructure.

**Theorem 38.** *Let  $K$  be a set of positive integers containing 3 and a positive even integer. Let  $K$  be a subset of  $\mathbb{N}_{0,1(3)}$  and  $\max\{k \in K\} \leq 22$ . Then exactly one of the following three cases is true:*

- (i) *There is no solution of system  $(\widetilde{S1})$ .*
- (ii) *Every solution of system  $(\widetilde{S1})$  is forbidden by a proposition given in Section 3.3.*
- (iii) *There exists a  $PBD[v, K]$ .*

### 5.3. The closure of sets containing 3 and an even integer and not all elements lie in $\mathbb{N}_{0,1(3)}$

**Lemma 39.**  $32 \notin B(\{3, 6, 11\})$ .



**Proof.** Since  $32 \notin B(\{3, 6\})$  we have at least one block of length 11, say  $B$ . Points from  $V \setminus B$  are all of type  $3^8 6^3$  or  $3^{13} 6^1$  and therefore  $B$  is the unique block of size 11. Proposition 26 implies that  $m' = 3$  and hence  $\beta_{K'} \leq 16$ ; together with Example 18 we obtain  $\beta_{K'} = 11$  or 12 with unique solutions  $(92, 11, 1; 0, 15, 0, 17)$ ,  $(87, 12, 1; 0, 12, 0, 20)$ , respectively. Note that, since there are at most 12 blocks of size 6, every point on  $B$  is of type  $3^8 6^1 11^1$ . Exactly  $88 = 8 \times 11$  blocks of length 3 intersect  $B$ , hence  $\beta_{K'} = 12$  is not possible. If  $\beta_{K'} = 11$ , then there are four 3-blocks which do not intersect  $B$ . Moreover, every point on these four blocks is of type  $3^{13} 6^1$ , since every block which contains a point of type  $3^8 6^3$  has a point in common with  $B$ . But four blocks of length 3 cover at least  $3 + 2 + 1 = 6$  points, contradicting that there are exactly  $15 - 11 = 4$  points of type  $3^{13} 6^1$ .  $\square$

**Lemma 40.**  $32 \notin B(\{3, 6, 14\})$  and  $35 \notin B(\{3, 6, 14\})$ .

**Proof.** Suppose first that  $v = 32$ . There exists exactly one solution of  $(\widetilde{S1})$ :  $(120, 3, 1; 0, 32, 0, 0)$ . The block of length 14 intersects  $14 \times 9 = 126$  blocks of length 3, a contradiction.

If  $v = 35$ , then the following point types are possible:  $3^4 14^2$ ,  $3^3 6^3 14^1$ ,  $3^8 6^1 14^1$ ,  $3^{12} 6^2$  and  $3^{17}$ . Inequality (6) of  $(\widetilde{S1})$  implies that  $\beta_{K'} \geq 15$ . Every point not on the unique 14-block that lies on a 6-block has point type  $3^{12} 6^2$ . There are  $\beta_{K'} - 1$  blocks of size 6 which cover at least  $(\beta_{K'} - 1)(6 - 1)/2 \geq 35$  points of this type. But  $35 + 14 > 35$ , a contradiction.  $\square$

**Lemma 41.**  $44 \notin B(\{3, 6, 17\})$ .

**Proof.** Let  $B$  be a block of length 17 in a  $PBD[44, \{3, 6, 17\}]$ . Every point outside of  $B$  has point type  $3^{14} 6^3$  or  $3^{19} 6^1$ . There are at least 17 blocks of size 6, since every point must be on at least one block of even length. These blocks cover at least  $17(6 - 1)/3 > 28$  points outside of  $B$ . But  $28 + 17 > 44$ , a contradiction.  $\square$

**Lemma 42.**  $44 \notin B(\{3, 6, 20\})$  and  $47 \notin B(\{3, 6, 20\})$ .

**Proof.** Let  $B$  be a block of length 20 in a  $PBD[44, \{3, 6, 20\}]$ . There is only one possible point type outside of  $B$ :  $3^{19} 6^1$ . Since five blocks of size 6 would cover more than 24 points there are exactly four 6-blocks, which are disjoint to  $B$ . Points on  $B$  have type  $3^{12} 20^1$  which implies that  $B$  intersects  $20 \times 12 = 240$  3-blocks. But according to Lemma 1 there are exactly 232 blocks of length 3.

If  $v = 47$ , then there are two point types possible for points from  $V \setminus B$ :  $3^{18} 6^2$  and  $3^{23}$ . Thus, no point outside of  $B$  lies on more than two blocks of length 6.  $B$  intersects at least 20 blocks of length 6, which cover at least  $20 \times 5/2 = 50$  points, a contradiction.  $\square$

**Lemma 43.**  $62 \notin B(\{3, 10, 20\})$ .

**Proof.** There exists only one solution of system  $(\widetilde{S1})$ :  $(292, 5, 1; 0, 58, 0, 4)$ . Assume the existence of a  $PBD[62, \{3, 10, 20\}]$  with a block of length 20, say  $B$ . Then point types  $3^{17}10^3$  and  $3^{26}10^1$  are possible outside of  $B$ . Suppose there exists a point of type  $3^{17}10^3$ , then every block through this point intersects  $B$ . Through each of these three intersection points goes one other 10-block, but there is no solution of  $(\widetilde{S1})$  with six 10-blocks. On the other hand there are now four points on  $B$  (of type  $3^{12}10^220^1$ ) which lie on eight distinct blocks of length 10, again a contradiction.  $\square$

**Theorem 44.** *Let  $K$  be a set of positive integers containing 3 and a positive even integer. Let  $K$  be not a subset of  $\mathbb{N}_{0,1(3)}$  and  $\max\{k \in K\} \leq 22$ . Then exactly one of the following four cases is true:*

- (i) *There is no solution of system  $(\widetilde{S1})$ .*
- (ii) *Every solution of system  $(S1)$  is forbidden by a proposition given in Subsection 3.3.*
- (iii) *There exists no  $PBD[v, K]$  by a lemma from this subsection.*
- (iv) *There exists a  $PBD[v, K]$ .*

### Appendix A. Statistics

Here we provide some statistics on the usage of the theorems and lemmas in determining the nonexistence of  $PBD$ s. A total of 18 773 possible parameter sets of  $PBD$ s were disallowed by the use of one of the statements. All necessary calculations are done on a 166 MHz personal computer in less than 3 min.

Most of the possible parameter sets (16 129) were eliminated by the use of Lemma 5 (Drake and Larson). Next, in 2578 cases there was no solution of system  $(\widetilde{S1})$  and thus no  $PBD$  (see Theorem 20). Note that Theorem 20 combines the results of Theorem 9, Corollary 10, Propositions 17, 19 and 26.

Single solutions of  $(\widetilde{S1})$  were ruled out for 12 possible parameter sets by Proposition 24 and in 19 cases by Proposition 25. Proposition 29 was applied to remove 21 solutions and Proposition 31 eliminated 7 solutions. The remaining 7 parameter sets are considered in Section 5.3.

Moreover, 837 solutions of  $(\widetilde{S1})$  were used to compute suitable  $PBD$ s as ingredients for recursive construction methods. In all but two cases the construction of the pre-structure was made automatically by a computer program. For the two exceptions see Examples 35 and 15. Here we used the dual of certain  $GDD$ s. The construction part took approximately 36 h on a PC.

### Appendix B. Table of closures

We present here the closure of subsets  $K \subset \{3, 4, \dots, 22\}$  with  $3 \in K$ ,  $K \cap \{11, 12, \dots, 22\} \neq \emptyset$  and the cardinality of  $K$  is at most three. Table 1 with complete results (the closures of all subsets of  $\{3, 4, \dots, 22\}$  containing 3) is located at

<ftp://ftp.math.uni-rostock.de/pub/members/mgruttm/closure.htm>

We do not mention the closure of  $K$  if  $B(K) = B(K \setminus \{k\})$  and  $k \in K$ .

Table 1

Subset	Closure	Genuine exceptions
3 11	$N_{1(2)}$	5 17
3 12	$N_{0,1(3)}$	4 6 10 16 18 22 24 28 30 40 42 46 52 54
3 14	N	4 5 6 8 10 11 12 16 17 18 20 22 23 24 26 28 29 30 32 34 35 36 38 41 44 46 47 48 50 52 53 54 58 59 60 62 64 65 66 68 70 71 72 74 77 80 83 88 89 90 94 95 96 100 101 104 107 110
3 16	$N_{0,1(3)}$	4 6 10 12 18 22 24 28 30 34 36 40 42 52 54 58 60 66 70 72 82 84
3 17	$N_{1(2)}$	5 11 23 29
3 18	$N_{0,1(3)}$	4 6 10 12 16 22 24 28 30 34 36 40 42 46 48 58 60 64 66 70 76 78 82 84 94 96
3 20	N	4 5 6 8 10 11 12 14 16 17 18 22 23 24 26 28 29 30 32 34 35 36 38 40 41 42 44 46 47 48 50 52 53 54 56 59 62 64 65 66 68 70 71 72 74 76 77 78 82 83 84 86 88 89 90 92 94 95 96 98 100 101 102 104 106 107 108 110 113 116 119 122 124 125 126 130 131 132 136 137 138 142 143 144 146 148 149 150 152 154 155 158 161 164 167 170 173 179 184 185 186 190 191 192 196 197 198 203 206 209 215
3 22	$N_{0,1(3)}$	4 6 10 12 16 18 24 28 30 34 36 40 42 46 48 52 54 58 60 70 72 76 78 82 84 90 94 96 100 102 112 114 118 120 136 138 156
3 4 11	N	5 6 8 14 17 20
3 4 14	N	5 6 8 11 17 20 23 26 29
3 4 17	N	5 6 8 11 14 20 23 26 29 32
3 4 20	N	5 6 8 11 14 17 23 26 29 32 35 38 41
3 5 12	N	4 6 8 10 14 16 18 20 22 24 26 28 30 32 38 40 42 44 46 50 52 54 62
3 5 14	N	4 6 8 10 12 16 18 20 22 24 26 28 30 32 34 36 38 44 46 48 50 52 54 58 60 62 64 72 74
3 5 16	N	4 6 8 10 12 14 18 20 22 24 26 28 30 32 34 36 38 40 42 44 50 52 54 56 58 60 62 66 68 70 72 74 82 84 86
3 5 18	N	4 6 8 10 12 14 16 20 22 24 26 28 30 32 34 36 38 40 42 44 46 48 50 56 58 60 62 64 66 68 70 74 76 78 80 82 84 92 94 96 98 110
3 5 20	N	4 6 8 10 12 14 16 18 22 24 26 28 30 32 34 36 38 40 42 44 46 48 50 52 54 56 62 64 66 68 70 72 74 76 78 82 84 86 88 90 92 94 102 104 106 108 110 122 124 142
3 5 22	N	4 6 8 10 12 14 16 18 20 24 26 28 30 32 34 36 38 40 42 44 46 48 50 52 54 56 58 60 62 68 70 72 74 76 78 80 82 84 86 90 92 94 96 98 100 102 104 112 114 116 118 120 122 134 136 138 156 158
3 6 11	N	4 5 8 10 12 14 17 20 22 26 32
3 6 12	$N_{0,1(3)}$	4 10 22
3 6 14	N	4 5 8 10 11 12 17 20 22 23 26 29 32 35

Table 1 continued

Subset	Closure	Genuine exceptions
3 6 17	N	4 5 8 10 11 12 14 20 22 23 26 29 32 38 44
3 6 20	N	4 5 8 10 11 12 14 17 22 23 26 29 32 35 38 41 44 47
3 6 22	$N_{0,1(3)}$	4 10 12
3 8 11	N	4 5 6 10 12 14 16 17 18 20 26 28 30 34 36 38
3 8 12	N	4 5 6 10 11 14 16 17 18 20 23 26 28 29 30 35 38
3 8 14	N	4 5 6 10 11 12 16 17 18 20 23 26 28 29 30 34 35 36
3 8 16	N	4 5 6 10 11 12 14 17 18 20 23 26 28 29 30 34 35 36 38
3 8 17	N	4 5 6 10 11 12 14 16 18 20 23 26 28 29 30 34 36 38
3 8 18	N	4 5 6 10 11 12 14 16 17 20 23 26 28 29 30 34 35 36 38
3 8 20	N	4 5 6 10 11 12 14 16 17 18 23 26 28 29 30 34 35 36 38
3 10 11	N	4 5 6 8 12 14 16 17 18 20 22 24 26 32 34 36 38 42 44
50 56 62 68		
3 10 12	$N_{0,1(3)}$	4 6 16 18 22 24
3 10 14	N	4 5 6 8 11 12 16 17 18 20 22 23 24 26 29 32 34 35 36
38 41 47 53 59 65		
3 10 16	$N_{0,1(3)}$	4 6 12 18 22 24 34 36 42
3 10 17	N	4 5 6 8 11 12 14 16 18 20 22 23 24 26 29 32 34 36 38
42 44 50 56 62 68 74		
3 10 18	$N_{0,1(3)}$	4 6 12 16 22 24 34 36 42
3 10 20	N	4 5 6 8 11 12 14 16 17 18 22 23 24 26 29 32 34 35 36
38 41 42 44 47 53 59 62 65 71 77 83		
3 10 22	$N_{0,1(3)}$	4 6 12 16 18 24 34 36 42
3 11 12	N	4 5 6 8 10 14 16 17 18 20 22 24 26 28 30 32 38 40 42
44 46 50 52 54 56 62 68 74 80 86 92		
3 11 14	N	4 5 6 8 10 12 16 17 18 20 22 24 26 28 30 32 34 36 38
44 46 48 50 52 54 58 60 62 64 66 68 70 72 74 80 88 90 94 96 100 104 110		
3 11 16	N	4 5 6 8 10 12 14 17 18 20 22 24 26 28 30 32 34 36 38
40 42 44 50 52 54 56 58 60 62 66 68 70 72 74 80 82 84 86 92 98 104 110 116		
122 128 134 140		
3 11 17	$N_{1(2)}$	5
3 11 18	N	4 5 6 8 10 12 14 16 17 20 22 24 26 28 30 32 34 36 38
40 42 44 46 48 50 56 58 60 62 64 66 68 70 74 76 78 80 82 84 86 92 94 96 98		
104 110 116 122 128 134 140 146 152 158		
3 11 20	N	4 5 6 8 10 12 14 16 17 18 22 24 26 28 30 32 34 36 38
40 42 44 46 48 50 52 54 56 62 64 66 68 70 72 74 76 78 82 84 86 88 90 92 94 96		
98 100 102 104 106 108 110 116 122 124 126 130 132 136 138 142 144 146 148		
150 152 154 158 164 170 184		
3 11 22	N	4 5 6 8 10 12 14 16 17 18 20 24 26 28 30 32 34 36 38
40 42 44 46 48 50 52 54 56 58 60 62 68 70 72 74 76 78 80 82 84 86 90 92 94 96		
98 100 102 104 110 112 114 116 118 120 122 128 134 136 138 140 146 152 156		
158 164 170 176 182 188 194 200 206		
3 12 14	N	4 5 6 8 10 11 16 17 18 20 22 23 24 26 28 29 30 32 35
38 41 44 46 47 52 53 59 65 71 77 83		
3 12 16	$N_{0,1(3)}$	4 6 10 18 22 24 28 30 40 42 54

Table 1 continued

Subset	Closure	Genuine exceptions
$\boxed{3\ 12\ 17}$	$\boxed{\text{N}}$	4 5 6 8 10 11 14 16 18 20 22 23 24 26 28 29 30 32 38 40 42 44 46 50 52 54 56 62 68 74 80 86 92 98 104
$\boxed{3\ 12\ 18}$	$\boxed{\text{N}_{0,1(3)}}$	4 6 10 16 22 24 28 30 40 42 46
$\boxed{3\ 12\ 20}$	$\boxed{\text{N}}$	4 5 6 8 10 11 14 16 17 18 22 23 24 26 28 29 30 32 35 38 40 41 42 44 46 47 50 52 53 54 59 62 65 71 77 83 89 95 101 107
$\boxed{3\ 12\ 22}$	$\boxed{\text{N}_{0,1(3)}}$	4 6 10 16 18 24 28 30 40 42 46 52 54
$\boxed{3\ 14\ 16}$	$\boxed{\text{N}}$	4 5 6 8 10 11 12 17 18 20 22 23 24 26 28 29 30 32 34 35 36 38 41 44 47 50 52 53 54 59 60 65 66 71 77 83 89 95 101 107
$\boxed{3\ 14\ 17}$	$\boxed{\text{N}}$	4 5 6 8 10 11 12 16 18 20 22 23 24 26 28 29 30 32 34 36 38 44 46 48 50 52 54 58 60 62 64 66 68 70 72 74 80 88 90 94 96 100 104 110
$\boxed{3\ 14\ 18}$	$\boxed{\text{N}}$	4 5 6 8 10 11 12 16 17 20 22 23 24 26 28 29 30 32 34 35 36 38 41 44 46 47 48 50 53 58 59 62 64 65 66 70 71 77 80 83 89 95 101 107
$\boxed{3\ 14\ 20}$	$\boxed{\text{N}}$	4 5 6 8 10 11 12 16 17 18 22 23 24 26 28 29 30 32 34 35 36 38 41 44 46 47 48 50 52 53 54 59 64 65 66 68 70 71 72 77 83 89 95 101 107
$\boxed{3\ 14\ 22}$	$\boxed{\text{N}}$	4 5 6 8 10 11 12 16 17 18 20 23 24 26 28 29 30 32 34 35 36 38 41 44 46 47 48 50 52 53 54 58 59 60 62 65 68 70 71 72 77 83 89 90 95 101 107
$\boxed{3\ 16\ 17}$	$\boxed{\text{N}}$	4 5 6 8 10 11 12 14 18 20 22 23 24 26 28 29 30 32 34 36 38 40 42 44 50 52 54 56 58 60 62 66 68 70 72 74 80 82 84 86 92 98 104 110 116 122 128 134 140 146 152 158 164 170 176
$\boxed{3\ 16\ 18}$	$\boxed{\text{N}_{0,1(3)}}$	4 6 10 12 22 24 28 30 34 36 40 42 58 60
$\boxed{3\ 16\ 20}$	$\boxed{\text{N}}$	4 5 6 8 10 11 12 14 17 18 22 23 24 26 28 29 30 32 34 35 36 38 40 41 42 44 47 50 52 53 54 56 59 62 65 66 70 71 72 74 77 82 83 84 86 89 95 101 107 113 119 125 131 137 143 149 155
$\boxed{3\ 16\ 22}$	$\boxed{\text{N}_{0,1(3)}}$	4 6 10 12 18 24 28 30 34 36 40 42 52 54 58 60 72
$\boxed{3\ 17\ 18}$	$\boxed{\text{N}}$	4 5 6 8 10 11 12 14 16 20 22 23 24 26 28 29 30 32 34 36 38 40 42 44 46 48 50 56 58 60 62 64 66 68 70 74 76 78 80 82 84 86 92 94 96 98 104 110 116 122 128 134 140 146 152 158 164 170 176 182 188 194 200 206 212
$\boxed{3\ 17\ 20}$	$\boxed{\text{N}}$	4 5 6 8 10 11 12 14 16 18 22 23 24 26 28 29 30 32 34 36 38 40 42 44 46 48 50 52 54 56 62 64 66 68 70 72 74 76 78 82 84 86 88 90 92 94 96 98 100 102 104 106 108 110 116 122 124 126 130 132 136 138 142 144 146 148 150 152 154 158 164 170 184 186 190 192 196 198 206
$\boxed{3\ 17\ 22}$	$\boxed{\text{N}}$	4 5 6 8 10 11 12 14 16 18 20 23 24 26 28 29 30 32 34 36 38 40 42 44 46 48 50 52 54 56 58 60 62 68 70 72 74 76 78 80 82 84 86 90 92 94 96 98 100 102 104 110 112 114 116 118 120 122 128 134 136 138 140 146 152 156 158 164 170 176 182 188 194 200 206 212 218 224 230 236 242 248 254 260 266 272 278 284
$\boxed{3\ 18\ 20}$	$\boxed{\text{N}}$	4 5 6 8 10 11 12 14 16 17 22 23 24 26 28 29 30 32 34 35 36 38 40 41 42 44 46 47 48 50 53 56 59 62 64 65 66 68 70 71 76 77 82 83 84 86 89 95 101 107 113 119 125 131 137 143 149 155 161 167 173 179
$\boxed{3\ 18\ 22}$	$\boxed{\text{N}_{0,1(3)}}$	4 6 10 12 16 24 28 30 34 36 40 42 46 48 58 60 70 78 82
$\boxed{3\ 20\ 22}$	$\boxed{\text{N}}$	4 5 6 8 10 11 12 14 16 17 18 23 24 26 28 29 30 32 34 35 36 38 40 41 42 44 46 47 48 50 52 53 54 56 59 62 65 68 70 71 72 74 76 77 78 83 84 89 90 92 94 95 96 101 107 110 113 119 125 131 137 143 149 155 161 167 173 179 185 191 197 203 209 215

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